Solution to Class Exercise 4

1. Find the volume of the region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$ and the surface $z = \cos(\pi x/2),\ 0 \le x \le 1$.

   Solution. This exercise is taken from 15.5 in Text. The volume is given by

$$\iint_D \int_0^{\cos(\pi x/2)} dV \ ,$$

   where D is the triangle bounded by the coordinates axes and $y = 1 - x$. Therefore, the volume is

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dzdydx &= \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) dydx \\
&= \int_0^1 (1-x)\cos\left(\frac{\pi x}{2}\right) dx \\
&= \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - \frac{2}{\pi}\left\{\left[x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - \int_0^1 \sin\left(\frac{\pi x}{2}\right) dx\right\} \\
&= \frac{2}{\pi} - \frac{2}{\pi}\left\{1 + \left[\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1\right\} \\
&= \frac{2}{\pi} - \frac{2}{\pi}\left\{1 - \frac{2}{\pi}\right\} \\
&= \frac{4}{\pi^2}
\end{aligned}$$

2. Find

$$\iiint_P dV \ ,$$

   where P is the solid whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.

   Solution. This exercise is taken from 15.7 in Text. Using cylindrical coordinates, the volume of P is given by

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} \int_0^{3-r\sin\theta} r\,dzdrd\theta &= \int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} (3 - r\sin\theta) r\,drd\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\frac{3r^2}{2} - \frac{r^3\sin\theta}{3}\right]_{\cos\theta}^{2\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\frac{9\cos^2\theta}{2} - \frac{7\cos^3\theta\sin\theta}{3}\right] d\theta \\
&= \left[\frac{9\theta}{4} + \frac{9\sin 2\theta}{8} + \frac{7\cos^4\theta}{12}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{9\pi}{4}
\end{aligned}$$